B. Sc. Part-II

Energy in Electromagnetic fields.

(Poynting Theorem)

From Maxwell's equation it is possible to derive an important expression which we shall recognise as the energy principle in an electromagnetic field.

For this consider Maxwell's equations (C) and (D) ie Ampere's and Faraday's Laws in differential forms.

$$\text{Curl}\, H = J + \frac{\partial D}{\partial t}$$

$$\text{and} \quad \text{Curl}\, E = -\frac{\partial B}{\partial t}$$

If we take the scaler product of equation (1) with E and of equation (2) with (−H) we get

$$E \cdot \text{Curl}(H) = E \cdot J + E \cdot \frac{\partial D}{\partial t}$$

and

$$-H \cdot \text{Curl}\, E = +H \cdot \frac{\partial D}{\partial t}$$

adding equation (3) and (4) we get

$$-H \cdot \text{Curl}\, E + E \cdot \text{Curl}\, H = J \cdot E + \left[E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t}\right]$$

But by the vector identity

$$H \cdot \text{Curl}\, E - E \cdot \text{Curl}\, H = \text{div}(E \times H)$$

The above equation reduces to

$$-\operatorname{div}(E \times H) = J \cdot E + \left[E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t}\right]$$

Now as $E \cdot \frac{\partial D}{\partial t} = \epsilon_r \epsilon_0 E \frac{\partial E}{\partial t} = \frac{1}{2} \epsilon_r \epsilon_0 \frac{\partial}{\partial t}(E \cdot E)$

$$= \frac{1}{2} \frac{\partial}{\partial t}(E \cdot D)$$

and $H \cdot \frac{\partial B}{\partial t} = \mu_r \mu_0 H \cdot \frac{\partial H}{\partial t}$

$$= \frac{1}{2} \mu \mu_0 \frac{\partial}{\partial t}(H \cdot H) = \frac{1}{2} \frac{\partial}{\partial t}(H \cdot B)$$

So equation (5) reduces to

$$J \cdot E + \frac{1}{2} \frac{\partial}{\partial t}(E \cdot D + H \cdot B) + \operatorname{div}(E \times H) = 0$$

Each term in the above equation can be given some physical meaning if it is multiplied by an element of volume dr and integrated over surface is S.

Thus the result is

$$\int_r (J \cdot E)\, dr + \int_r \frac{1}{2} \frac{\partial}{\partial t}(E \cdot D + H \cdot B)\, dr + \int_r \operatorname{div}(E \times H)\, dr = 0$$

But as $\int_r \operatorname{div}(E \times H)\, dr = \oint_S (E \times H) \cdot ds$

So. $\int_r (J \cdot E)\, dr + \int_r \frac{1}{2} \frac{\partial}{\partial t}(E \cdot D + H \cdot B)\, dr$

$$+ \oint_s (E \times H) \cdot ds = 0 \qquad (A)$$

To understand what equation means,

Let us ~~know~~ now interpret various term in it

(A) Interperation of $\int_V J \cdot E \, dV$

The current distribution represented by the J can be considered as made up of various charges q_i moving with velocity v_i so that

$$\int J \cdot E \, dV = \int I \, dl \cdot E$$

$$= \int dq \, v \cdot E$$

$$= \sum q_i (v_i \cdot E_i) \qquad (7)$$

Where E_i denotes the electric field at the position of charge q_i.

Now the electromagnetic force on the ith charged particle is given by the Lorentz expression

$$F_i = q_i (E_i + v_i \times B)$$

So the work done per unit time on the charge q_i by the field will be

$$\frac{\partial W_i}{\partial t} = F_i \cdot V_i \qquad \left[\frac{dW}{dt} = \frac{F \cdot dl}{dt} = F \cdot v\right]$$

$$= q_i (E_i + V_i \times B_i) \cdot V_i \qquad (\text{as } F_i = q_i (E_i + V_i \times B_i))$$

ie $$\frac{\partial W_i}{\partial t} = q_i V_i \cdot E_i \quad [\text{as } V_i \cdot (V_i \times B_i) = (V_i \times V_i) \cdot B_i = 0]$$

So the rate at which the work is done by the field on the charges is

$$\frac{\partial W}{\partial t} = \sum \frac{\partial W_i}{\partial t}$$

$$= \sum q_i \cdot V_i \cdot E_i \qquad \text{———— (8)}$$

Comparing equation (7) and (8) denotes

$$\int J \cdot E \, dr = \frac{dW}{dt}$$

ie the first term $\int (J \cdot E)\, dr$ represents the rate at which work is done by the field on the charges.

Continue